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# Analysis of the viscous Cahn–Hilliard equation in $\mathbb{R}^N$

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Solvability of Cauchy's problem in $\mathbb{R}^N$ for an extended viscous Cahn–Hilliard equation is studied. The problem is considered first in a standard Sobolev space $H^1(\mathbb{R}^N)$, next a notion of the '$H$-solution' is introduced that is well adapted to the structure of the viscous Cahn–Hilliard equation. Several properties of the solutions are reported, in particular those connected with their asymptotic behavior. We collect here also known properties of the unbounded operator $(-\Delta)^{-1}$ in $\mathbb{R}^N$ that are needed in our considerations.



## 1. Introduction

The classical Cahn–Hilliard model introduced in 1958 is one of the most often studied problem of mathematical physics. It has also many extensions and generalizations (see e.g. [1,20,10]), among them the *viscous Cahn–Hilliard equation* that was introduced by A. Novick-Cohen in [21] to analyze the dynamics of the first order phase transitions. The viscous Cahn–Hilliard equation

$$(1-\nu)u_t = -\Delta\big(\Delta u + \bar{f}(u) - \nu u_t\big), \quad \text{in } \Omega, \tag{1}$$

where $\nu \in [0,1]$ and $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, includes as limiting cases the Cahn–Hilliard equation (when $\nu = 0$) and semilinear heat equation (when $\nu = 1$) (see [8,12,15]). The transition of the asymptotic behavior, as parameter $\nu$ varies from 0 to 1, was studied in [9] and later in [4].

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The dynamics of the viscous Cahn–Hilliard equation considered in bounded domains is quite well understood nowadays (see e.g. [1,21,28,9,4]). A rather natural extension is to study that problem in (space) unbounded domains $\Omega$. The first study of a variant of the original Cahn–Hilliard equation in $\mathbb{R}^N$, including a priori estimates in $L^\infty(\mathbb{R}^N)$ and existence of the global solutions, was reported in [3]. The case of unbounded domains in which the Poincaré inequality still holds (compare [26]) was studied later in [2] and [22], while considerations related to global existence of solutions corresponding to particular ('close to a constant') initial data were reported in [18]. Note that Eq. (1) in $\mathbb{R}^N$ has a rich set of evident solutions; it is satisfied by an arbitrary constant function $u(t,x)=c$.

This paper is devoted to the global solvability and properties of solutions to the Cauchy problem in $\mathbb{R}^N$, $N \geqslant 3$, for the extended *viscous Cahn–Hilliard equation*

$$\begin{cases} (1-\nu)u_t = -\Delta\big(\Delta u + f(x,u) - \nu u_t\big), & t>0,\ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases} \tag{2}$$

where $\nu \in [0,1)$ and the assumptions on the nonlinear function $f$ will be stated later.

If we decide to stay inside the classical Sobolev spaces setting, a serious problem is that the spectrum of the negative Laplacian $-\Delta$ in $\mathbb{R}^N$ equals $[0,\infty)$, and is purely absolutely continuous (see [7, p. 4]). That means, minus Laplacian in $\mathbb{R}^N$ is invertible but $(-\Delta)^{-1}$ is an unbounded operator while its domain is dense in $L^2(\mathbb{R}^N)$. Since in order to study Eq. (2) we need usually to invert the first right-hand side (minus) Laplacian in (2), we are thus forced to work with such an operator $(-\Delta)^{-1}$; some of its properties are recalled in Appendix A. In [6] a modification of (2) was proposed where we replaced the operator $(-\Delta)$ with $(-\Delta+\epsilon I)$, $\epsilon>0$ fixed. Local and global solvability and asymptotic behavior for such modification was reported in that paper. Now we want to consider analogous questions for the original problem (2).

We study first local solvability of (2) in the standard phase space $H^1(\mathbb{R}^N)$. Due to the technical difficulties, when obtaining asymptotic estimates of the $H^1(\mathbb{R}^N)$-solutions, we were forced to consider a smaller phase space; we called the solutions belonging to it '$H$-solutions' (that class is preserved in time). It is then much easier to estimate the $H$-solutions starting from one of the 'inverted' form of the problem (2); Eq. (40) or (43). This approach appears to be very well adapted to the structure of the problem under consideration. Thus, the most involved results of that paper are reported in Section 4 devoted to the $H$-solutions. In general, this paper provides a technical background for the further studies of the viscous Cahn–Hilliard equation in the whole $\mathbb{R}^N$. Furthermore, $Q(\cdot), Q'(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ denote the generic continuously increasing functions, and $C$, $c$ are the generic positive constants.

## 2. Setting of the problem and its local solvability

Throughout the paper the space dimension $N$ is taken greater or equal 3 and less or equal 6 (due to the limitation imposed in Assumption II). The standard notation of the Sobolev spaces is used and by $a^-$ ($a\in\mathbb{R}$) we denote any number strictly less than $a$, but close to $a$.

### 2.1. Operator $A_\nu$

Let $\nu\in(0,1)$. Denote by $A_\nu$ the operator $((1-\nu)I-\nu\Delta)$ in $L^2(\mathbb{R}^N)$ with the domain $D(A_\nu)=H^2(\mathbb{R}^N)$. It is known that $A_\nu$, $\nu\in(0,1)$, is a closed positive definite and self-adjoint operator in $L^2(\mathbb{R}^N)$ with $\sigma(A_\nu)\subset[1-\nu,+\infty)$, so that $(A_\nu)^{-1}\in\mathcal{L}(L^2(\mathbb{R}^N))$ for $\nu\in(0,1)$. It is also known (e.g. [13, p. 32]) that the operator $-\Delta$ is sectorial in $L^p(\mathbb{R}^N)$, $1\leqslant p<\infty$; in particular it is closable in $L^p(\mathbb{R}^N)$.

### 2.2. Formulation of the problem and its local solvability

We are interested in real valued solutions of the problem (2). Our nonlinearity will have the form:

$$f(x,u) = f_1(x,u) - \alpha u,$$

where $\alpha > 0$ and the function $f_1$ satisfies Assumptions I and II below. In particular the function

$$f_1(x,u) = \beta(x)u - \gamma u^3,$$

where $\beta \in L^N(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is such that $\frac{\partial \beta}{\partial x_i} \in L^N(\mathbb{R}^N)$ and $\frac{\partial^2 \beta}{\partial x_i^2} \in L^{\frac{N+6}{4}}(\mathbb{R}^N)$, satisfies all our assumptions when $N = 3$. However, this function growth too fast when $N > 3$, i.e. the assumption (3) is not satisfied.

Our first task is the local in time solvability of the problem (2) in the phase space $H^1(\mathbb{R}^N)$. For the local solvability of (2) we need to impose the following conditions on the nonlinear term (compare with [24,6] for similar assumptions):

**Assumption I.**

- The function $f_1 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous with $f_1(\cdot, 0) \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and satisfies the local Lipschitz condition:

$$\exists_{q \in [0, \frac{4}{N-2})} \exists_{L>0} \forall_{s_1, s_2 \in \mathbb{R}} \forall_{x \in \mathbb{R}^N} \quad \bigl|f_1(x,s_1) - f_1(x,s_2)\bigr| \leqslant L|s_1 - s_2|\bigl(I(x) + |s_1|^q + |s_2|^q\bigr), \tag{3}$$

  where we take $0 \leqslant I(\cdot) \in L^N(\mathbb{R}^N)$.
- Denoting $F_1(x,s) = \int_0^s f_1(x,z)\,dz$ the antiderivative (primitive) of $f_1$, the following estimate holds:

$$\exists_{0<\mu<\frac{\alpha}{2}} \exists_{0 \leqslant C_\mu(\cdot) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)} \exists_{0 \leqslant \phi(\cdot) \in L^1(\mathbb{R}^N)} \forall_{x \in \mathbb{R}^N,\, s \in \mathbb{R}} \quad F_1(x,s) \leqslant \mu s^2 + C_\mu(x)|s| + \phi(x). \tag{4}$$

Consider thus the viscous Cahn–Hilliard equation

$$\begin{cases} (1-\nu)u_t = -\Delta\bigl(\Delta u + f(x,u) - \nu u_t\bigr), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases} \tag{5}$$

where $\nu \in (0,1)$. With the use of the operator $A_\nu$ the problem (5) considered on the phase space $H^1(\mathbb{R}^N)$ will be rewritten as

$$\begin{aligned} u_t &= A_\nu^{-1}\bigl(-\Delta^2 u - \Delta f(x,u)\bigr) \\ &= -\left(\frac{1}{\nu^2}A_\nu - \frac{2-2\nu}{\nu^2}I + \frac{(1-\nu)^2}{\nu^2}A_\nu^{-1}\right)u + \left(\frac{1}{\nu}I - \frac{1-\nu}{\nu}A_\nu^{-1}\right)f(x,u), \end{aligned}$$

$$u(0) = u_0 \in H^1\bigl(\mathbb{R}^N\bigr). \tag{6}$$

The last expression shows that the viscous Cahn–Hilliard equation will be viewed as an equation with sectorial operator:

$$u_t = -B_\nu u + \left(\frac{1}{\nu}I - \frac{1-\nu}{\nu}A_\nu^{-1}\right)f(x,u) = -B_\nu u + \mathcal{F}(u),$$

where $B_\nu = (\frac{1}{\nu^2}A_\nu - \frac{2-2\nu}{\nu^2}I + \frac{(1-\nu)^2}{\nu^2}A_\nu^{-1})$ is a bounded perturbation of the sectorial positive operator $\frac{1}{\nu^2}A_\nu$, so $B_\nu$, $\nu \in (0,1)$, is also sectorial (see [13, Corollary 1.4.5]). We will show that with Assumption I, for $\nu \in (0,1)$, the nonlinearity $\mathcal{F}(u) = (\frac{1}{\nu}I - \frac{1-\nu}{\nu}A_\nu^{-1})f(\cdot,u)$ acting from $H^1(\mathbb{R}^N)$ into $H^{-1+\epsilon}(\mathbb{R}^N)$ is Lipschitz continuous on bounded sets.

Let $N \geqslant 3$ and $0 < \epsilon \ll 1$. Fix a bounded set $B \subset H^1(\mathbb{R}^N)$ and let $u_1, u_2 \in B$. Since $H^{-1+\epsilon}(\mathbb{R}^N) \supset L^m(\mathbb{R}^N)$ for $m \in [\frac{2N}{N+2-2\epsilon}, 2]$, due to assumption (3), we will write

$$\begin{aligned}
&\left\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\right\|_{H^{-1+\epsilon}(\mathbb{R}^N)} \\
&\quad \leqslant C\left(\left\||u_1 - u_2|\left(I(\cdot) + |u_1|^q + |u_2|^q\right)\right\|_{L^{m^*}(\mathbb{R}^N)} + \|u_1 - u_2\|_{L^2(\mathbb{R}^N)}\right),
\end{aligned} \tag{7}$$

where $m^* = \frac{2}{q+1}$ for $0 \leqslant q < \frac{2-2\epsilon}{N}$ and $m^* = \frac{2N}{N+2-2\epsilon}$ for $\frac{2-2\epsilon}{N} \leqslant q \leqslant \frac{4-2\epsilon}{N-2}$. Then, thanks to the embedding $H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ for $p \in [2, \frac{2N}{N-2}]$, and the Hölder inequality, we get

$$\left\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\right\|_{H^{-1+\epsilon}(\mathbb{R}^N)} \leqslant C(B)\|u_1 - u_2\|_{H^1(\mathbb{R}^N)}.$$

Thus, for $\nu \in (0,1)$, the local solution of (5) varying in the phase space $H^1(\mathbb{R}^N)$ exists (see [13]). Note, that the exponent $\alpha$ in the scale of fractional order spaces connected with the operators $A_\nu$ is given through:

$$X^\alpha = H^1\left(\mathbb{R}^N\right), \qquad X = H^{-1+\epsilon}\left(\mathbb{R}^N\right),$$

so that $\alpha = \frac{2-\epsilon}{2} < 1$, and Henry's approach [13] applies. The local solution $u$ satisfies:

$$\begin{aligned}
u \in C\left([0, \tau_{\max}); H^1\left(\mathbb{R}^N\right)\right) \cap C\left((0, \tau_{\max}); H^{1+\epsilon}\left(\mathbb{R}^N\right)\right),& \\
u_t \in C\left((0, \tau_{\max}); H^{(1+\epsilon)^-}\left(\mathbb{R}^N\right)\right),&
\end{aligned} \tag{8}$$

where $\tau_{\max}$ is the *'life time'* of the local solution $u$ and $a^-$ denotes any real number strictly less than $a$. Moreover, the solution fulfills Cauchy's integral formula:

$$u(t) = e^{-B_\nu t}u_0 + \int_0^t e^{-B_\nu(t-s)}\mathcal{F}\left(u(s)\right)ds, \tag{9}$$

where $e^{-B_\nu t}$ is the linear analytic semigroup corresponding to the operator $B_\nu$ in $H^{-1+\epsilon}(\mathbb{R}^N)$.

Consider next the case $\nu = 0$. As for $\nu \in (0,1)$ we want the phase space to be equal to $H^1(\mathbb{R}^N)$. Eq. (5) with $\nu = 0$ will be then satisfied in the space $H^{-3+\epsilon}(\mathbb{R}^N)$, and $\mathcal{F}(u) = -\Delta(f(\cdot,u))$. The Lipschitz condition from bounded subset of $X_0^{\alpha'} = H^1(\mathbb{R}^N)$ to $X_0 = H^{-3+\epsilon}(\mathbb{R}^N)$ ($\alpha' = \frac{4-\epsilon}{4}$) will be easily verified in that case, since:

$$\left\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\right\|_{X_0} = \left\|-\Delta\left(f(\cdot,u_1) - f(\cdot,u_2)\right)\right\|_{X_0} \leqslant c\left\|f(\cdot,u_1) - f(\cdot,u_2)\right\|_X,$$

and further estimate will be given precisely as in (7). So, local solvability in $H^1(\mathbb{R}^N)$ is also justified for $\nu = 0$. The smoothness of $u$ is the same as in (8), and a version of the Cauchy formula is also available.

### 2.3. Higher regularity of the local solutions

In further studies we will need higher regularity of the local $H^1(\mathbb{R}^N)$-solutions. Such a property is natural for the solutions of parabolic problems with subcritical nonlinearity and will be obtained using the *bootstrapping argument*. The solution starting at $t = 0$ in $u_0 \in H^1(\mathbb{R}^N)$ will enter, for $t > 0$, $H^{1+\epsilon}(\mathbb{R}^N)$ (due to (8); we discuss here $\nu \in (0,1)$). Hence, for arbitrary small $t_0 > 0$ we will consider a solution to (5) with initial data $u(t_0) \in H^{1+\epsilon}(\mathbb{R}^N)$. After a finite repetition of that procedure, if $f$ is sufficiently smooth, we will obtain an arbitrary finite $H^k$-regularity of the solution $u(t)$ for $t > 0$ and as long as it exists (since $t_0$ was an arbitrarily small positive number).

Due to regularization of the local $H^1(\mathbb{R}^N)$-solution, proceeding as in [13,5] and using (9), we have an explicit estimate in $H^{1+\epsilon^-}(\mathbb{R}^N)$ (here we denote $0 < \delta := \epsilon - \epsilon^- \ll 1$)

$$\begin{aligned}\|u(t)\|_{H^{1+\epsilon^-}(\mathbb{R}^N)} &\leqslant \|e^{-B_\nu t}\|_{\mathcal{L}(H^1(\mathbb{R}^N),H^{1+\epsilon^-}(\mathbb{R}^N))}\|u_0\|_{H^1(\mathbb{R}^N)} \\ &\quad + \int_0^t \|e^{-B_\nu (t-s)}\|_{\mathcal{L}(H^{-1+\epsilon}(\mathbb{R}^N),H^{1+\epsilon^-}(\mathbb{R}^N))}\|\mathcal{F}(u(s))\|_{H^{-1+\epsilon}(\mathbb{R}^N)}\,ds \\ &\leqslant \frac{c_\nu}{t^{\frac{\epsilon^-}{2}}}\|u_0\|_{H^1(\mathbb{R}^N)} + \int_0^t \frac{c_\nu}{(t-s)^{\frac{2-\delta}{2}}}\|\mathcal{F}(u(s))\|_{H^{-1+\epsilon}(\mathbb{R}^N)}\,ds.\end{aligned}$$

When $f \in C^2$ and fulfills Assumption II below, single use of the regularization procedure described above will show that $u(t) \in H^3(\mathbb{R}^N)$ for $t > 0$ as long as the local solution exists.

**Assumption II.** We assume that $3 \leqslant N \leqslant 6$, and $f \in C^2$ with

$$\begin{aligned}&\forall_{\eta>0}\ \exists_{0\leqslant C_\eta(\cdot)\in L^N(\mathbb{R}^N)}\ \forall_{s\in\mathbb{R}}\ \forall_{x\in\mathbb{R}^N}\quad \left|\frac{\partial^2 f_1}{\partial x_i \partial s}(x,s)\right| \leqslant \eta|s|^{\frac{6}{N-2}} + C_\eta(x), \\ &\forall_{\eta>0}\ \exists_{0\leqslant \tilde{C}_\eta(\cdot)\in L^\infty(\mathbb{R}^N)}\ \forall_{s\in\mathbb{R}}\ \forall_{x\in\mathbb{R}^N}\quad \left|\frac{\partial^2 f_1}{\partial s^2}(x,s)\right| \leqslant \eta|s|^{\frac{6-N}{N-2}} + \tilde{C}_\eta(x), \\ &\forall_{\eta>0}\ \exists_{0\leqslant \bar{C}_\eta(\cdot)\in L^2(\mathbb{R}^N)}\ \forall_{s\in\mathbb{R}}\ \forall_{x\in\mathbb{R}^N}\quad \left|\frac{\partial^2 f_1}{\partial x_i^2}(x,s)\right| \leqslant \eta|s|^{\frac{N+6}{N-2}} + \bar{C}_\eta(x),\end{aligned} \tag{10}$$

moreover, $\frac{\partial f_1}{\partial s}(\cdot,0) \in L^\infty(\mathbb{R}^N)$ and $\frac{\partial f_1}{\partial x_i}(\cdot,0) \in L^2(\mathbb{R}^N)$.

When $N = 6$ we need additionally to take $\eta = 0$ in the second condition (the critical exponent for $f_1$ equals 2 in that case). This implies that the second derivative $\frac{\partial^2 f_1}{\partial s^2}$ is bounded by $\|\tilde{C}_0\|_{L^\infty(\mathbb{R}^N)}$. Moreover we need to assume that the function $\tilde{C}_0$ is such that $\|\tilde{C}_0\|_{L^\infty(\mathbb{R}^N)}$ is sufficiently small.

**Remark 1.** As a consequence of the first and second assumptions in (10) we get

$$\left|\frac{\partial f_1}{\partial x_i}(x,s)\right| \leqslant \eta|s|^{\frac{N+4}{N-2}} + C_\eta(x)|s| + \left|\frac{\partial f_1}{\partial x_i}(x,0)\right|, \tag{11}$$

$$\left|\frac{\partial f_1}{\partial s}(x,s)\right| \leqslant \eta|s|^{\frac{4}{N-2}} + \tilde{C}_\eta(x)|s| + \left|\frac{\partial f_1}{\partial s}(x,0)\right|, \tag{12}$$

respectively.

The estimates of the higher norms of the solutions will be given in the more complicated case $N = 3, 4, 5$. The calculations for the case $N = 6$ are contained in the presented one.

**Lemma 1.** *When $f$ fulfills Assumption* II *then the local $H^1(\mathbb{R}^N)$-solution is bounded, for $t > 0$ and as long as it exists, in $H^2(\mathbb{R}^N)$ and $H^3(\mathbb{R}^N)$ as stated explicitly in* (16) *and* (25) *below.*

**Proof.** Note that multiplying (5) by $\Delta u$ we get

$$\int_{\mathbb{R}^N} \big((1-\nu)I - \nu\Delta\big)u_t \Delta u \, dx = \int_{\mathbb{R}^N} |\nabla\Delta u|^2 \, dx + \int_{\mathbb{R}^N} \nabla f(x,u)\cdot\nabla\Delta u \, dx.$$

Using (11) and (12) we have that

$$\begin{aligned}
\big\|\nabla f(\cdot,u)\big\|_{L^2(\mathbb{R}^N)} &\leqslant \left\|\frac{\partial f_1}{\partial u}(\cdot,u)|\nabla u|\right\|_{L^2(\mathbb{R}^N)} + \alpha\|\nabla u\|_{L^2(\mathbb{R}^N)} + \left\|\frac{\partial f_1}{\partial x_i}(\cdot,u)\right\|_{L^2(\mathbb{R}^N)} \\
&\leqslant c\left(\eta\|u\|^{\frac{4}{N-2}}_{L^\infty(\mathbb{R}^N)} + \|\tilde{C}_\eta\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^\infty(\mathbb{R}^N)} + \left\|\frac{\partial f_1}{\partial u}(\cdot,0)\right\|_{L^\infty(\mathbb{R}^N)} + 1\right)\|\nabla u\|_{L^2(\mathbb{R}^N)} \\
&\quad + c\left(\eta\|u\|^{\frac{4+N}{N-2}}_{L^{\frac{8+2N}{N-2}}(\mathbb{R}^N)} + \|C_\eta\|_{L^N(\mathbb{R}^N)}\|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} + \left\|\frac{\partial f_1}{\partial x_i}(\cdot,0)\right\|_{L^2(\mathbb{R}^N)}\right).
\end{aligned}$$

Now it follows from Agmon's inequality, a Sobolev embedding and interpolation that, if $N \leqslant 6$, there is a constant $C > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C\big(\|\nabla\Delta u\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}\big)^\alpha \|u\|^{1-\alpha}_{H^1(\mathbb{R}^N)}, \quad \frac{N-2}{4} \leqslant \alpha \leqslant 1, \tag{13}$$

$$\|u\|_{L^{\frac{8+2N}{N-2}}(\mathbb{R}^N)} \leqslant c\big(\|\nabla\Delta u\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}\big)^{\frac{N-2}{N+4}} \|u\|^{\frac{6}{N+4}}_{H^1(\mathbb{R}^N)}. \tag{14}$$

If $N = 3, 4, 5$, we choose $\alpha = \frac{N-2}{4}$ in (13), and the above estimates together with the bound of the local solution in $H^1(\mathbb{R}^N)$ (valid for $t \in [0, \tau^-_{\max}]$) lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left((1-\nu)\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \nu\int_{\mathbb{R}^N} (\Delta u)^2 \, dx\right) dx \\
&\quad \leqslant -\|\nabla\Delta u\|^2_{L^2(\mathbb{R}^N)} + c'\big(\eta\|\nabla\Delta u\|^2_{L^2(\mathbb{R}^N)} + const\big(\|u_0\|_{H^1(\mathbb{R}^N)}\big)\big).
\end{aligned} \tag{15}$$

Note that we can add and subtract to the right-hand side of (15) the term $\frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}$. The 'plus' term will be next incorporated to $const(\|u_0\|_{H^1(\mathbb{R}^N)})$ (in case of the local solutions it is bounded in terms of $\|u_0\|_{H^1(\mathbb{R}^N)}$, while for global in time solutions we use the uniform in time estimate (30) to get the same conclusion). Therefore, the right-hand side of (15) can be extended to

$$\begin{aligned}
&-\|\nabla\Delta u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + c''\big(\eta\|\nabla\Delta u\|^2_{L^2(\mathbb{R}^N)} + const\big(\|u_0\|_{H^1(\mathbb{R}^N)}\big)\big) \\
&\quad \leqslant -\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + const\big(\|u_0\|_{H^1(\mathbb{R}^N)}\big)
\end{aligned}$$

for sufficiently small $\eta > 0$, where an elementary estimate has also been used

$$\int_{\mathbb{R}^N} (\Delta u)^2 \, dx = -\int_{\mathbb{R}^N} \nabla\Delta u \cdot \nabla u \, dx \leqslant \|\nabla\Delta u\|_{L^2(\mathbb{R}^N)}\|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

Setting $\mathcal{L}_1(\phi) := (1-\nu)\int_{\mathbb{R}^N} |\nabla\phi|^2\,dx + \nu\int_{\mathbb{R}^N} (\Delta\phi)^2\,dx$, the final estimate implies that

$$\frac{d}{dt}\mathcal{L}_1\big(u(t,\cdot)\big) \leqslant -c\mathcal{L}_1\big(u(t,\cdot)\big) + const'\big(\|u_0\|_{H^1(\mathbb{R}^N)}\big), \tag{16}$$

with $const'(\|u_0\|_{H^1(\mathbb{R}^N)})$ independent of $\nu$.

We proceed with the estimates of better norms of solutions to (2). Multiplying (5) by $(-\Delta)^2 u$ we get

$$\frac{(1-\nu)}{2}\,\frac{d}{dt}\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \frac{\nu}{2}\,\frac{d}{dt}\|\nabla\Delta u\|^2_{L^2(\mathbb{R}^N)} = -\big\|\Delta^2 u\big\|^2_{L^2(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N} \Delta f(x,u)\Delta^2 u\,dx. \tag{17}$$

We need to estimate explicitly the last component

$$\big\|\Delta f(x,u)\big\|_{L^2(\mathbb{R}^N)} = \left\| \sum_{i=1}^N \frac{\partial^2 f_1}{\partial x_i^2} + 2\sum_{i=1}^N \frac{\partial^2 f_1}{\partial x_i\partial u}\frac{\partial u}{\partial x_i} + |\nabla u|^2\frac{\partial^2 f_1}{\partial u^2} + \left(\frac{\partial f_1}{\partial u} - \alpha\right)\Delta u \right\|_{L^2(\mathbb{R}^N)}, \tag{18}$$

where, thanks to (10) and interpolation inequalities, the right-hand side terms will be estimated one by one as follows:

$$\left\|\frac{\partial^2 f_1}{\partial x_i^2}\right\|_{L^2(\mathbb{R}^N)} \leqslant c\big(\eta\|u\|^{\frac{N+6}{N-2}}_{L^{\frac{2N+12}{N-2}}(\mathbb{R}^N)} + \|\bar{C}_\eta\|_{L^2(\mathbb{R}^N)}\big) \leqslant c\big(\|u\|_{H^1(\mathbb{R}^N)}\big)\big(\eta\|u\|_{H^4(\mathbb{R}^N)} + 1\big). \tag{19}$$

Next

$$\begin{aligned}\left\|\frac{\partial^2 f_1}{\partial x_i\partial u}\frac{\partial u}{\partial x_i}\right\|_{L^2(\mathbb{R}^N)} &\leqslant c\left\|\frac{\partial^2 f_1}{\partial x_i\partial u}\right\|_{L^N(\mathbb{R}^N)}\|\nabla u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \\ &\leqslant c\big(\eta\|u\|^{\frac{6}{N-2}}_{L^{\frac{6N}{N-2}}(\mathbb{R}^N)} + \|C_\eta\|_{L^N(\mathbb{R}^N)}\big)\|\nabla u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)},\end{aligned}$$

but

$$\|u\|^{\frac{6}{N-2}}_{L^{\frac{6N}{N-2}}(\mathbb{R}^N)} \leqslant c\|u\|^{\frac{22-2N}{3N-6}}_{H^1(\mathbb{R}^N)}\|u\|^{\frac{2}{3}}_{H^4(\mathbb{R}^N)},$$

and

$$\|u\|_{W^{1,\frac{2N}{N-2}}(\mathbb{R}^N)} \leqslant c\|u\|^{\frac{2}{3}}_{H^1(\mathbb{R}^N)}\|u\|^{\frac{1}{3}}_{H^4(\mathbb{R}^N)},$$

hence

$$\left\|\frac{\partial^2 f_1}{\partial x_i\partial u}\frac{\partial u}{\partial x_i}\right\|_{L^2(\mathbb{R}^N)} \leqslant Q\left(\|u\|_{H^1(\mathbb{R}^N)}\right)\big(\eta\|u\|_{H^4(\mathbb{R}^N)} + \|C_\eta\|_{L^N(\mathbb{R}^N)}\|u\|^{\frac{1}{3}}_{H^4(\mathbb{R}^N)}\big). \tag{20}$$

Since

$$\|\Delta u\|_{L^2(\mathbb{R}^N)} \leqslant c\|u\|_{H^1(\mathbb{R}^N)}^{\frac{2}{3}}\|u\|_{H^4(\mathbb{R}^N)}^{\frac{1}{3}},$$
$$\|\nabla u\|_{L^4(\mathbb{R}^N)}^2 \leqslant c\|u\|_{H^1(\mathbb{R}^N)}^{\frac{12-N}{6}}\|u\|_{H^4(\mathbb{R}^N)}^{\frac{N}{6}},$$

and

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C\|u\|_{H^1(\mathbb{R}^N)}^{1-\theta}\|u\|_{H^4(\mathbb{R}^N)}^{\theta}, \quad \frac{N-2}{6} \leqslant \theta < 1,$$

we have

$$\begin{aligned}\left\||\nabla u|^2\frac{\partial^2 f_1}{\partial u^2}\right\|_{L^2(\mathbb{R}^N)} &\leqslant \left\|\frac{\partial^2 f_1}{\partial u^2}\right\|_{L^\infty(\mathbb{R}^N)}\big\||\nabla u|\big\|_{L^4(\mathbb{R}^N)}^2 \leqslant c\big(\eta\|u\|_{L^\infty(\mathbb{R}^N)}^{\frac{6-N}{N-2}} + \|\tilde{C}_\eta\|_{L^\infty(\mathbb{R}^N)}\big)\|\nabla u\|_{L^4(\mathbb{R}^N)}^2 \\ &\leqslant Q\left(\|u\|_{H^1(\mathbb{R}^N)}\right)\big(\eta\|u\|_{H^4(\mathbb{R}^N)} + \|\tilde{C}_\eta\|_{L^\infty(\mathbb{R}^N)}\|u\|_{H^4(\mathbb{R}^N)}^{\frac{N}{6}}\big),\end{aligned} \tag{21}$$

and also

$$\begin{aligned}&\left\|\left(\frac{\partial f_1}{\partial u} - \alpha\right)\Delta u\right\|_{L^2(\mathbb{R}^N)} \\ &\quad\leqslant \left(\left\|\frac{\partial f_1}{\partial u}\right\|_{L^\infty(\mathbb{R}^N)} + \alpha\right)\|\Delta u\|_{L^2(\mathbb{R}^N)} \\ &\quad\leqslant c\left(\eta\|u\|_{L^\infty(\mathbb{R}^N)}^{\frac{4}{N-2}} + \|\tilde{C}_\eta\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^\infty(\mathbb{R}^N)} + \left\|\frac{\partial f_1}{\partial u}(\cdot,0)\right\|_{L^\infty(\mathbb{R}^N)} + 1\right)\|u\|_{H^1(\mathbb{R}^N)}^{\frac{2}{3}}\|u\|_{H^4(\mathbb{R}^N)}^{\frac{1}{3}} \\ &\quad\leqslant Q\left(\|u\|_{H^1(\mathbb{R}^N)}\right)\left(\eta\|u\|_{H^4(\mathbb{R}^N)} + \|\tilde{C}_\eta\|_{L^\infty(\mathbb{R}^N)}\|u\|_{H^4(\mathbb{R}^N)}^{\frac{N}{6}}\right. \\ &\qquad\left. + \left(\left\|\frac{\partial f_1}{\partial u}(\cdot,0)\right\|_{L^\infty(\mathbb{R}^N)} + 1\right)\|u\|_{H^4(\mathbb{R}^N)}^{\frac{1}{3}}\right).\end{aligned} \tag{22}$$

Collecting (18)–(22) we finally get

$$\begin{aligned}\big\|\Delta f(x,u)\big\|_{L^2(\mathbb{R}^N)} &\leqslant Q\left(\|u\|_{H^1(\mathbb{R}^N)}\right)\big(\eta\|u\|_{H^4(\mathbb{R}^N)} + 1\big) \\ &\leqslant Q'\big(\|u\|_{H^1(\mathbb{R}^N)}\big)\big(\eta\big\|\Delta^2 u\big\|_{L^2(\mathbb{R}^N)} + 1\big).\end{aligned} \tag{23}$$

The last estimate together with (17) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big((1-\nu)\|\Delta u\|_{L^2(\mathbb{R}^N)}^2 + \nu\|\nabla\Delta u\|_{L^2(\mathbb{R}^N)}^2\big) \\ &\quad\leqslant -\big\|\Delta^2 u\big\|_{L^2(\Omega)}^2 + c\big(\eta\big\|\Delta^2 u\big\|_{L^2(\Omega)}^2 + const\big(\|u_0\|_{H^1(\mathbb{R}^N)}\big)\big).\end{aligned} \tag{24}$$

Setting $\mathcal{L}_2(\phi) := (1-\nu)\int_{\mathbb{R}^N}(\Delta\phi)^2\,dx + \nu\int_{\mathbb{R}^N}|\nabla\Delta\phi|^2\,dx$, for sufficiently small $\eta > 0$ we have that

$$\frac{d}{dt}\mathcal{L}_2\big(u(t,\cdot)\big) \leqslant -c\mathcal{L}_2\big(u(t,\cdot)\big) + const\big(\|u_0\|_{H^1(\mathbb{R}^N)}\big), \tag{25}$$

with $const(\|u_0\|_{H^1(\mathbb{R}^N)})$ independent of $\nu$. $\square$

## 3. Global solvability of (5)

### 3.1. A priori estimates

We will present next the, uniform in time, a priori estimate of the solutions to (5) in $H^1(\mathbb{R}^N)$. Such an estimate is sufficient to extend the local solution in the phase space $H^1(\mathbb{R}^N)$ globally in time.

**Remark 2.** As a consequence of the assumption (3) we get

$$\left|f_1(x,s)\right| \leqslant LI(x)|s| + L|s|^{q+1} + \left|f_1(x,0)\right|. \tag{26}$$

The last condition provides us the following estimate on the primitive $F_1$:

$$\left|F_1(x,s)\right| \leqslant \int_0^{|s|} \left|f_1(x,z)\right| dz \leqslant \frac{L}{2} I(x)|s|^2 + \frac{L}{q+2}|s|^{q+2} + \left|f_1(x,0)\right||s|. \tag{27}$$

But we usually prefer to use a slightly different condition (4).

To get the global in time a priori estimate in $H^1(\mathbb{R}^N)$ we multiply Eq. (5) by the *potential* $\omega := \Delta u + f(x,u) - \nu u_t$ and integrate the result over $\mathbb{R}^N$

$$(1-\nu)\int_{\mathbb{R}^N} \omega u_t \, dx = \int_{\mathbb{R}^N} |\nabla\omega|^2 \, dx \geqslant 0. \tag{28}$$

Note that, as a consequence of (8) and (25), the expression $|\nabla\omega|$ is well defined for $t > 0$ and as long as the local $H^1(\mathbb{R}^N)$-solution exists. Consequently, we find that

$$\frac{d}{dt}\left(\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \alpha\|u\|^2_{L^2(\mathbb{R}^N)} - 2\int_{\mathbb{R}^N} F_1(x,u)\,dx\right) + \nu(1-\nu)\|u_t\|^2 \leqslant 0. \tag{29}$$

Then, thanks to (4) and (27), we obtain

$$\begin{aligned}
&\left\|\nabla u(t)\right\|^2_{L^2(\mathbb{R}^N)} + (\alpha - 2\mu - \zeta)\left\|u(t)\right\|^2_{L^2(\mathbb{R}^N)} \\
&\quad \leqslant \|\nabla u_0\|^2_{L^2(\mathbb{R}^N)} + \alpha\|u_0\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\zeta}\|C_\mu\|^2_{L^2(\mathbb{R}^N)} + 2\|\phi\|_{L^1(\mathbb{R}^N)} + L\|I\|_{L^N(\mathbb{R}^N)}\|u_0\|^2_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)} \\
&\qquad + C\|u_0\|^{q+2}_{H^1(\mathbb{R}^N)} + 2\|u_0\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}\left\|f_1(\cdot,0)\right\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)},
\end{aligned} \tag{30}$$

where the constant $\zeta$ is chosen such that $M_1 := (\alpha - 2\mu - \zeta) > 0$.

**Remark 3.** For the future needs it is important to note that the above $H^1(\mathbb{R}^N)$ a priori estimate of the solution $u(t)$ is uniform for $t > 0$ and also for $u_0$ varying in (centered at zero) balls $B(0,\rho) \subset H^1(\mathbb{R}^N)$. It follows also from (30) that images of bounded subsets of $H^1(\mathbb{R}^N)$ through the semigroup generated by (5) on $H^1(\mathbb{R}^N)$ stay bounded for $t > 0$.

We will show that, as a consequence of the above uniform in time a priori estimate of $u$ in $H^1(\mathbb{R}^N)$, such a local solution will be extended globally in time in the phase space $H^1(\mathbb{R}^N)$. We will follow the idea of [13, Corollary 3.3.5, p. 56] extended in [5, Chapter 3], under the name of *subordination condition* to prove this property. We need to show the estimate (here $\nu \in (0,1)$):

$$\begin{aligned}\|\mathcal{F}(u(t))\|_{H^{-1+\epsilon}(\mathbb{R}^N)} &= \left\| \left(\frac{1}{\nu} I - \frac{1-\nu}{\nu} A_\nu^{-1}\right) f(\cdot, u(t)) \right\|_{H^{-1+\epsilon}(\mathbb{R}^N)} \\ &\leqslant K\left(\|u(t)\|_{H^1(\mathbb{R}^N)}\right), \end{aligned} \tag{31}$$

with a nondecreasing function $K:[0,\infty) \to [0,\infty)$ (which is a simple form of the subordination condition). We only mention that the estimate of the $H^{-1+\epsilon}(\mathbb{R}^N)$ norm through the $H^1(\mathbb{R}^N)$ norm follows closely the estimate (31). Consequently we obtain a subordination allowing us to extend the local $H^1(\mathbb{R}^N)$-solution globally in time.

The case $\nu = 0$ should be treated separately. The considerations concerning $H^1(\mathbb{R}^N)$ a priori estimate (30) stay valid for $\nu = 0$ as well. The subordination condition now reads:

$$\begin{aligned}\|\mathcal{F}(u(t))\|_{H^{-3+\epsilon}(\mathbb{R}^N)} &= \|-\Delta(f(\cdot, u(t)))\|_{H^{-3+\epsilon}(\mathbb{R}^N)} \\ &\leqslant K''\left(\|u(t)\|_{H^1(\mathbb{R}^N)}, \|f(\cdot,0)\|_{L^2(\mathbb{R}^N)}\right), \end{aligned} \tag{32}$$

as in the proof of (31). Consequently, global in time extendibility of the local solution $u(t)$ in $H^1(\mathbb{R}^N)$ follows.

Hence the problem (5), $\nu \in [0,1)$, defines on the phase space $H^1(\mathbb{R}^N)$ a semigroup:

$$S(t)u_0 = u(t), \quad t \geqslant 0, \tag{33}$$

where $u(t)$ is the global in time $H^1(\mathbb{R}^N)$-solution corresponding to initial datum $u_0$ constructed above.

### 3.2. Lyapunov function

We will show that the expression below is a Lyapunov function for the problem (5), more precisely $L: H^1(\mathbb{R}^N) \to \mathbb{R}$ with

$$L(u) = \alpha \|u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)} - 2 \int_{\mathbb{R}^N} F_1(x,u)\,dx, \tag{34}$$

where the function $F_1(x,u) = \int_0^u f_1(x,s)\,ds$ is a primitive of $f_1$.

With our assumptions (3), (4), the following properties of $L$ are easy to obtain:

- $L$ is bounded from below.
- $L$ is continuous on $H^1(\mathbb{R}^N)$.
- For each $u \in H^1(\mathbb{R}^N)$ the function $(0,+\infty) \ni t \mapsto L(S(t)u) \in \mathbb{R}$ is nonincreasing.

It follows from (29) that

$$\frac{d}{dt} L(u(t)) \leqslant 0.$$

- For any $v \in H^1(\mathbb{R}^N)$ we have

$$L(S(t)v) = const \quad \text{for all } t \geqslant 0 \quad \text{implies that} \quad S(t)v = v \quad \text{for all } t \geqslant 0.$$

We have (see (28)):

$$\frac{d}{dt}L\big(u(t)\big) = \frac{-2}{1-\nu}\int\limits_{\mathbb{R}^N} |\nabla\omega|^2\,dx - 2\nu\big\|u_t(t)\big\|^2_{L^2(\mathbb{R}^N)} \leqslant 0 \quad \text{for } t>0, \tag{35}$$

so that in particular, when the functional $L$ is constant on the solution $u(t)$,

$$\big\|u_t(t)\big\|^2_{L^2(\mathbb{R}^N)} = 0 \quad \text{for } t>0.$$

Consequently, $u_t(t,x) = 0$ a.e. in $\mathbb{R}^N$ for $t>0$.

- $L(\varphi) \to \infty$ as $\|\varphi\|_{H^1(\mathbb{R}^N)} \to \infty$.

Using (4) and the Cauchy inequality we obtain

$$\begin{aligned} L(u) &\geqslant \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + (\alpha - 2\mu - \zeta)\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{\zeta}\|C_\mu\|^2_{L^2(\mathbb{R}^N)} - 2\|\phi\|_{L^1(\mathbb{R}^N)} \\ &\geqslant \left(\min\{1, (\alpha-2\mu-\zeta)\}\|u\|^2_{H^1(\mathbb{R}^N)} - \frac{1}{\zeta}\|C_\mu\|^2_{L^2(\mathbb{R}^N)} - 2\|\phi\|_{L^1(\mathbb{R}^N)}\right) \to \infty \end{aligned} \tag{36}$$

as $\|u\|_{H^1(\mathbb{R}^N)} \to \infty$ (the constant $\zeta$ is chosen such that $\alpha - 2\mu - \zeta > 0$).

### 3.3. *Stationary solutions of the problem (5)*

To get an estimate of the stationary solution of (5) we need to assume the following condition:

$$\exists_{0<\delta<\alpha}\ \exists_{0\leqslant C_\delta(\cdot)\in L^2(\mathbb{R}^N)}\ \exists_{0\leqslant\psi(\cdot)\in L^1(\mathbb{R}^N)}\ \forall_{x\in\mathbb{R}^N,\, s\in\mathbb{R}} \quad f_1(x,s)s \leqslant \delta s^2 + C_\delta(x)|s| + \psi(x). \tag{37}$$

We present here an estimate, in $H^1(\mathbb{R}^N)$, of the stationary solutions $v$ of (5).

Note that the stationary solution $v$ is a weak solution of the problem

$$\Delta v + f(x,v) = 0, \quad x \in \mathbb{R}^N. \tag{38}$$

Multiplying (38) by $v$ and integrating over $\mathbb{R}^N$, due to (37), we get

$$\|\nabla v\|^2_{L^2(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} f(x,v)v\,dx \leqslant (\delta-\alpha)\|v\|^2_{L^2(\mathbb{R}^N)} + \|C_\delta\|_{L^2(\mathbb{R}^N)}\|v\|_{L^2(\mathbb{R}^N)} + \|\psi\|_{L^1(\mathbb{R}^N)}.$$

Then, using the Cauchy inequality, we have

$$\|\nabla v\|^2_{L^2(\mathbb{R}^N)} + (\alpha-\delta-\xi)\|v\|^2_{L^2(\mathbb{R}^N)} \leqslant \frac{1}{4\xi}\|C_\delta\|^2_{L^2(\mathbb{R}^N)} + \|\psi\|_{L^1(\mathbb{R}^N)}, \tag{39}$$

where the constant $\xi$ is such that $\alpha - \delta - \xi > 0$.

**Remark 4.** We have shown existence of a Lyapunov function for the problem (5). Also, under an extra assumption (37), the set of the stationary solutions is bounded in $H^1(\mathbb{R}^N)$. Consequently, the Cauchy problem (5) generates a **gradient system** in $H^1(\mathbb{R}^N)$ in the sense of [23]. In particular, boundedness in $H^1(\mathbb{R}^N)$ of the set of stationary solutions guarantees that the semigroup of the global $H^1(\mathbb{R}^N)$-solutions to (5) is *point dissipative* in $H^1(\mathbb{R}^N)$.

## 4. Restricted phase space

### 4.1. The phase space $H$ and its preliminary properties

We have already introduced and studied the global solutions to Cauchy's problem (2) in $H^1(\mathbb{R}^N)$. In the further considerations of the asymptotic behavior of solutions we face technical difficulties when proving asymptotic compactness of the semigroup generated by (2) on that phase space. It is easier to work on an 'inverted' form of Eq. (2), which reads

$$(1-\nu)(-\Delta)^{-1}u_t + \nu u_t = \Delta u + f(x,u), \tag{40}$$

and is obtained by applying the operator $(-\Delta)^{-1}$ to both sides of (2). Unfortunately, the operator $(-\Delta)^{-1}$ is unbounded and to be able to work on 'inverted' equation (40) we need to introduce a *subclass H* of the $H^1(\mathbb{R}^N)$-solutions (which will be shown to be preserved in time).

**Definition 1.** An $H^1(\mathbb{R}^N)$-solution to (5) will be called *H-solution* provided that $(-\Delta)^{-\frac{1}{2}}u_0 \in L^2(\mathbb{R}^N)$ and

$$(-\Delta)^{-\frac{1}{2}}u \in L^\infty\big(t_0,T;L^2\big(\mathbb{R}^N\big)\big) \quad \text{and} \quad (-\Delta)^{-1}u_t \in L^2\big(0,T;L^2\big(\mathbb{R}^N\big)\big) \tag{41}$$

for arbitrary $0 < t_0 < T$.

**Remark 5.** Following the general definition of the domain of a linear unbounded operator, we have

$$D\big((-\Delta)^{-\frac{1}{2}}\big) = \big\{\phi \in L^2\big(\mathbb{R}^N\big);\, (-\Delta)^{-\frac{1}{2}}\phi \in L^2\big(\mathbb{R}^N\big)\big\}.$$

Since in case of the $H$-solutions: $u, u_t, (-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-1}u_t \in L^2(\mathbb{R}^N)$ (at least for a.e. $t \in (0,T)$), then $u_t \in D((-\Delta)^{-1}) \subset D((-\Delta)^{-\frac{1}{2}})$. Consequently we have:

$$u, (-\Delta)^{-\frac{1}{2}}u_t \in D\big((-\Delta)^{-\frac{1}{2}}\big). \tag{42}$$

When proving the estimates of the $H$-solutions an 'intermediate' form of Eq. (2) will be frequently used, which reads

$$(1-\nu)(-\Delta)^{-\frac{1}{2}}u_t = (-\Delta)^{\frac{1}{2}}\big(\Delta u + f(x,u) - \nu u_t\big). \tag{43}$$

It is obtained by applying the operator $(-\Delta)^{-\frac{1}{2}}$ to (2).

A preliminary property of the $H$-solution is formulated next. We have namely:

**Lemma 2.** *A stronger requirement in* (41) *is true for the H-solutions, namely that* $(-\Delta)^{-\frac{1}{2}}u \in C((0,T); L^2(\mathbb{R}^N))$. *Moreover, formula* (46) *holds.*

**Proof.** It follows from [16, p. 299 and Problem 3.32, p. 279], that the realization of $(-\Delta)$ in $L^2(\mathbb{R}^N)$ is *m-accretive* in the sense of Kato. Consequently, by [16, Theorem 3.35, p. 281], there exists a unique *m-accretive* square root $(-\Delta)^{\frac{1}{2}}$, which is also self-adjoint and nonnegative.

Since (e.g. [19,17]) $((-\Delta)^\alpha)^{-1} = (-\Delta)^{-\alpha}$, $\alpha \in (0,1)$, and $(-\Delta)^{\frac{1}{2}}$ is symmetric with dense range

$$R\big((-\Delta)^{\frac{1}{2}}\big) = D\big((-\Delta)^{-\frac{1}{2}}\big) \supset L^{\frac{2N}{N+2}}\big(\mathbb{R}^N\big) \cap L^2\big(\mathbb{R}^N\big),$$

then, by [16, p. 269, Problem 3.11], $(-\Delta)^{-\frac{1}{2}}$ is symmetric, by Problem 3.30, p. 279 there, $(-\Delta)^{-\frac{1}{2}}$ is accretive.

We will show next that $(-\Delta)^{-\frac{1}{2}}u_t = ((-\Delta)^{-\frac{1}{2}}u)_t$. According to (8) $u_t \in C((0,\infty); H^{(1+\epsilon)^-}(\mathbb{R}^N))$, so that in particular for $|h| \leqslant h_0$, $t \in [t_0, T]$, $t_0 > 0$,

$$\left\| u_t(t) - \frac{u(t+h)-u(t)}{h} \right\|_{L^2(\mathbb{R}^N)} \leqslant \gamma\big(|h|\big) \to 0 \quad \text{as } |h| \to 0.$$

Now, for arbitrary $\eta \in \mathcal{S}$, due to Proposition 2,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}} u_t \eta \, dx - \frac{\int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}} u(t+h)\eta \, dx - \int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}} u(t)\eta \, dx}{h} \right| \\
&\quad = \left| \int_{\mathbb{R}^N} \left( (-\Delta)^{-\frac{1}{2}} u_t - \frac{(-\Delta)^{-\frac{1}{2}} u(t+h) - (-\Delta)^{-\frac{1}{2}} u(t)}{h} \right) \eta \, dx \right| \\
&\quad = \left| \int_{\mathbb{R}^N} \left( (-\Delta)^{-\frac{1}{2}} u_t - (-\Delta)^{-\frac{1}{2}} \frac{u(t+h)-u(t)}{h} \right) \eta \, dx \right| \\
&\quad \leqslant \left\| (-\Delta)^{-\frac{1}{2}} u_t - (-\Delta)^{-\frac{1}{2}} \frac{u(t+h)-u(t)}{h} \right\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \|\eta\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)} \\
&\quad \leqslant c \left\| u_t(t) - \frac{u(t+h)-u(t)}{h} \right\|_{L^2(\mathbb{R}^N)} \|\eta\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)} \\
&\quad \leqslant \gamma\big(|h|\big) \|\eta\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}.
\end{aligned} \tag{44}$$

The last estimate shows that

$$\frac{(-\Delta)^{-\frac{1}{2}} u(t+h) - (-\Delta)^{-\frac{1}{2}} u(t)}{h} \rightharpoonup (-\Delta)^{-\frac{1}{2}} u_t \quad \text{as } |h| \to 0,$$

weakly in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. Consequently, there exists a weak $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ derivative $((-\Delta)^{-\frac{1}{2}}u)_t$ (see e.g. [11, Chapter IV, 1]) and it is equal to $(-\Delta)^{-\frac{1}{2}}u_t$, which belongs to $L^2(t_0, T; L^2(\mathbb{R}^N))$ as assumed for the $H$-solution. Explicitly, thanks to (44),

$$\forall_{t \in [t_0,T]}\ \forall_{\eta \in \mathcal{S}} \quad \int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}} u_t \eta \, dx = \int_{\mathbb{R}^N} \big((-\Delta)^{-\frac{1}{2}} u\big)_t \eta \, dx = \frac{d}{dt} \int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}} u \eta \, dx. \tag{45}$$

It now follows from (41), (45) and denseness of $\mathcal{S}$ in $L^2(\mathbb{R}^N)$ that the condition (iii) of [27, p. 69], is satisfied for $(-\Delta)^{-\frac{1}{2}}u$ and $(-\Delta)^{-\frac{1}{2}}u_t$ (with $X = L^2(\mathbb{R}^N)$), so that $(-\Delta)^{-\frac{1}{2}}u$ is a primitive of $(-\Delta)^{-\frac{1}{2}}u_t$ according to [27, Lemma 3.1].

Finally, by the Lions lemma (e.g. [27, p. 71] with $V = H = L^2(\mathbb{R}^N)$) and the comment following (45), for $H$-solution $u$ the function $(-\Delta)^{-\frac{1}{2}}u(\cdot)$ is equivalent a.e. to a continuous function $[t_0, T] \to L^2(\mathbb{R}^N)$. Moreover, in the sense of scalar distributions,

$$2 \int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}} u_t (-\Delta)^{-\frac{1}{2}} u \, dx = \frac{d}{dt} \big\| (-\Delta)^{-\frac{1}{2}} u \big\|^2_{L^2(\mathbb{R}^N)} \tag{46}$$

on $(0, T)$, since $t_0 > 0$ was arbitrary. □

Due to the result of [19], in the space dimensions $N \geqslant 5$ one can use directly the inverted form of Eq. (40) in estimates of the $H$-solutions. The two equalities are satisfied in that case, as stated in the lemma below.

**Lemma 3.** *For the $H$-solutions, in space dimensions $N \geqslant 5$, the following two equalities are valid*:

$$\int_{\mathbb{R}^N} (-\Delta)^{-1} u_t u \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \left[(-\Delta)^{-\frac{1}{2}} u\right]^2 dx,$$
$$\int_{\mathbb{R}^N} (-\Delta)^{-1} u_t u_t \, dx = \int_{\mathbb{R}^N} \left[(-\Delta)^{-\frac{1}{2}} u_t\right]^2 dx. \tag{47}$$

**Proof.** We start by checking the first condition in (47). When $u_t \in L^2(\mathbb{R}^N)$ and $N \geqslant 5$, the equality $(-\Delta)^{-1} u_t = (-\Delta)^{-\frac{1}{2}} (-\Delta)^{-\frac{1}{2}} u_t$ follows from Corollary 1 (with $p = 2, \alpha = \beta = \frac{1}{2}$). Consequently,

$$\int_{\mathbb{R}^N} (-\Delta)^{-1} u_t u \, dx = \int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}} (-\Delta)^{-\frac{1}{2}} u_t u \, dx. \tag{48}$$

The rest of the proof is a direct repetition of the proof of Lemma 2.

The proof of the second condition in (47) is included in the proof of the first condition of (47). □

We next have:

**Lemma 4.** *Any $H^1(\mathbb{R}^N)$-solution to* (5) *with $(-\Delta)^{-\frac{1}{2}} u_0 \in L^2(\mathbb{R}^N)$ is actually an $H$-solution to* (5)*.*

**Proof.** We will check first condition (41) in part connected to $(-\Delta)^{-1} u_t$. Recalling Eq. (40):

$$(1 - \nu)(-\Delta)^{-1} u_t = -\nu u_t + \Delta u + f(x, u),$$

we observe that thanks to (8), $u_t \in L^2(0, T; L^2(\mathbb{R}^N))$. A similar property of the two further components at the right-hand side above can be shown as follows. We multiply (5) by $u$ to get

$$\frac{1}{2} \frac{d}{dt} \left( (1 - \nu) \int_{\mathbb{R}^N} u^2 \, dx + \nu \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) = -\int_{\mathbb{R}^N} (\Delta u)^2 \, dx - \int_{\mathbb{R}^N} f(x, u) \Delta u \, dx$$
$$\leqslant -\frac{1}{2} \int_{\mathbb{R}^N} (\Delta u)^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} f^2(x, u) \, dx. \tag{49}$$

Moreover, thanks to (26), we have that

$$\int_{\mathbb{R}^N} f^2(x, u) \, dx \leqslant c \left( \int_{\mathbb{R}^N} \left(I^2(x) + 1\right) u^2 \, dx + \int_{\mathbb{R}^N} |u|^{2(q+1)} \, dx + \int_{\mathbb{R}^N} f_1^2(0, x) \, dx \right). \tag{50}$$

Applying a version of the Nirenberg–Gagliardo inequality:

$$\|\phi\|_{L^{2(q+1)}(\mathbb{R}^N)}^{2(q+1)} \leqslant c \|\phi\|_{H^2(\mathbb{R}^N)}^{Nq-2(q+1)} \|\phi\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^{Nq} \tag{51}$$

(where, thanks to the restriction on $q$ in (3); $Nq - 2(q+1) < 1$) to the intermediate right-hand side component (50), due to (49), we conclude that $\Delta u \in L^2(0,T;L^2(\mathbb{R}^N))$. Consequently, $f(\cdot,u) \in L^2(0,T;L^2(\mathbb{R}^N))$. Therefore, $(-\Delta)^{-1}u_t \in L^2(0,T;L^2(\mathbb{R}^N))$. This implies, since $u_t \in L^2(0,T;L^2(\mathbb{R}^N))$ for the $H^1(\mathbb{R}^N)$-solutions, that $u_t \in D((-\Delta)^{-1}) \subset D((-\Delta)^{-\frac{1}{2}})$.

Similar observation for $(-\Delta)^{\frac{1}{2}}u$ follows, through usual Gronwall inequality, from the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big[(-\Delta)^{-\frac{1}{2}}u(t)\big]^2\,dx - \int_{\mathbb{R}^N} \big[(-\Delta)^{-\frac{1}{2}}u_0\big]^2\,dx \\
&\quad = \int_0^t \frac{d}{ds}\int_{\mathbb{R}^N} \big[(-\Delta)^{-\frac{1}{2}}u(s)\big]^2\,dx\,ds \\
&\quad = 2\int_0^t \int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}}u_s(s)(-\Delta)^{-\frac{1}{2}}u(s)\,dx\,ds \\
&\quad \leqslant \int_0^t \bigg(\int_{\mathbb{R}^N} \big[(-\Delta)^{-\frac{1}{2}}u(s)\big]^2\,dx + \int_{\mathbb{R}^N} \big[(-\Delta)^{-\frac{1}{2}}u_s(s)\big]^2\,dx\bigg)\,ds,
\end{aligned}$$

since $(-\Delta)^{-\frac{1}{2}}u_s \in L^2(0,T;L^2(\mathbb{R}^N))$. Consequently, $u \in D((-\Delta)^{-\frac{1}{2}})$.

Using next the assumption (37), multiplying (43) by $(-\Delta)^{-\frac{1}{2}}u$, we get an estimate:

$$\frac{1-\nu}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \big[(-\Delta)^{-\frac{1}{2}}u\big]^2\,dx + \frac{\nu}{2}\frac{d}{dt}\int_{\mathbb{R}^N} u^2\,dx = -\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} f(x,u)u\,dx. \tag{52}$$

This leads to another Lyapunov type functional $L_1(\phi) = \frac{1-\nu}{2}\int_{\mathbb{R}^N}[(-\Delta)^{-\frac{1}{2}}\phi]^2\,dx + \frac{\nu}{2}\int_{\mathbb{R}^N}\phi^2\,dx$, which satisfies

$$\frac{d}{dt}L_1\big(u(t)\big) + \int_{\mathbb{R}^N} |\nabla u|^2\,dx \leqslant (\delta - \alpha)\int_{\mathbb{R}^N} u^2\,dx + \int_{\mathbb{R}^N} C_\delta(x)|u|\,dx + \int_{\mathbb{R}^N} \psi(x)\,dx. \tag{53}$$

Consequently,

$$L_1\big(u(t)\big) \leqslant L_1(u_0) + c\int_0^t \int_{\mathbb{R}^N} \big(C_\delta^2(x) + \psi(x)\big)\,dx\,ds, \tag{54}$$

so that any $H^1(\mathbb{R}^N)$-solution with $(-\Delta)^{-\frac{1}{2}}u_0 \in L^2(\mathbb{R}^N)$ preserves such property in time

$$\frac{1-\nu}{2}\int_{\mathbb{R}^N} \big[(-\Delta)^{-\frac{1}{2}}u(t)\big]^2\,dx \leqslant L_1(u_0) + ct\int_{\mathbb{R}^N} \big(C_\delta^2(x) + \psi(x)\big)\,dx. \tag{55}$$

This proves that the first condition required in (41) is satisfied. $\square$

### 4.2. Structure of the phase space of the $H$-solutions

It is clear that for arbitrary $u_0 \in H^1(\mathbb{R}^N)$ with $(-\Delta)^{-\frac{1}{2}} u_0 \in L^2(\mathbb{R}^N)$ there exists a unique (since the $H^1(\mathbb{R}^N)$-solution is unique) $H$-solution to (5). Such a solution varies in the phase space $Y = \{\phi \in H^1(\mathbb{R}^N);\ (-\Delta)^{-\frac{1}{2}} \phi \in L^2(\mathbb{R}^N)\}$, a linear subspace of $H^1(\mathbb{R}^N)$. As a consequence of Proposition 2, it is also clear that

$$H^1\big(\mathbb{R}^N\big) \cap L^{\frac{2N}{N+2}}\big(\mathbb{R}^N\big) \subset Y \subset H^1\big(\mathbb{R}^N\big).$$

In particular $Y$ is dense in $H^1(\mathbb{R}^N)$. We can introduce in the set $Y$ a seminorm

$$|\phi|_{-\frac{1}{2}} = \big\|(-\Delta)^{-\frac{1}{2}} \phi\big\|_{L^2(\mathbb{R}^N)},$$

and consider $Y$ as a metric space normed by

$$\|\phi\|_Y = \|\phi\|_{H^1(\mathbb{R}^N)} + |\phi|_{-\frac{1}{2}}.$$

It can be seen that the space $Y$ equipped with the scalar product

$$(\phi, \psi)_{H^1(\mathbb{R}^N)} + \big((-\Delta)^{-\frac{1}{2}} \phi, (-\Delta)^{-\frac{1}{2}} \psi\big)_{L^2(\mathbb{R}^N)}$$

is a pre-Hilbert space (see [27, p. 55] for a quite similar comment).

### 4.3. Asymptotic a priori estimate of the $H$-solutions

We will show now existence of a bounded absorbing set for the semigroup $\{S(t)\}$. We multiply (43) by $(-\Delta)^{\frac{-1}{2}} u_t$ and integrate over $\mathbb{R}^N$. The right-hand side will be next calculated approximating $(-\Delta)^{\frac{-1}{2}} u_t$ in $L^2(\mathbb{R}^N)$ by a sequence $\{\phi_n\} \subset H^1(\mathbb{R}^N)$ (so that $(-\Delta)^{\frac{1}{2}} \phi_n \to u_t$ in $H^{-1}(\mathbb{R}^N)$), and using self-adjointness of $(-\Delta)^{\frac{1}{2}}$ in $L^2(\mathbb{R}^N)$. We get

$$\begin{aligned}(1-\nu)\big\|(-\Delta)^{-\frac{1}{2}} u_t\big\|^2_{L^2(\mathbb{R}^N)} + \nu \|u_t\|^2_{L^2(\mathbb{R}^N)} \\ = \frac{d}{dt}\left(-\frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} - \frac{\alpha}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} F_1(x,u)\,dx\right).\end{aligned} \tag{56}$$

Let

$$E(\phi) = \frac{1-\nu}{2}\big\|(-\Delta)^{-\frac{1}{2}} \phi\big\|^2_{L^2(\mathbb{R}^N)} + \frac{\alpha+\nu}{2}\|\phi\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|\nabla\phi\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F_1(x,\phi)\,dx \tag{57}$$

and

$$E_1(\phi) = -\|\nabla\phi\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f(x,\phi)\phi\,dx. \tag{58}$$

Adding (56) and (52), due to (47), (57) and (58), we obtain

$$\frac{d}{dt}E\big(u(t)\big) \leqslant E_1\big(u(t)\big),$$

and then

$$E\big(u(t)\big) \leqslant E(u_0) + \int_0^t E_1\big(u(s)\big)\,ds.$$

Moreover, from the assumptions (4) and (37), we get

$$\begin{aligned} E\big(u(t)\big) \geqslant \frac{1}{2}\big((\alpha+\nu-2\mu-\zeta)\big\|u(t)\big\|^2_{L^2(\mathbb{R}^N)} + \big\|\nabla u(t)\big\|^2_{L^2(\mathbb{R}^N)}\big) \\ - \left(\frac{1}{2\zeta}\|C_\mu\|^2_{L^2(\mathbb{R}^N)} + \|\phi\|_{L^1(\mathbb{R}^N)}\right) \end{aligned} \tag{59}$$

and

$$E_1\big(u(t)\big) \leqslant -\big\|\nabla u(t)\big\|^2_{L^2(\mathbb{R}^N)} - (\alpha-\delta-\xi)\big\|u(t)\big\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{4\xi}\|C_\delta\|^2_{L^2(\mathbb{R}^N)} + \|\psi\|_{L^1(\mathbb{R}^N)}, \tag{60}$$

where the constants $\zeta$ and $\xi$ are chosen such that $\alpha-\delta-\xi>0$ and $\alpha+\nu-2\mu-\zeta>0$. Consequently, we have

$$\begin{aligned} &\frac{1}{2}\big((\alpha+\nu-2\mu-\zeta)\big\|u(t)\big\|^2_{L^2(\mathbb{R}^N)} + \big\|\nabla u(t)\big\|^2_{L^2(\mathbb{R}^N)}\big) \\ &\quad\leqslant E(u_0) + \frac{1}{2\zeta}\|C_\mu\|^2_{L^2(\mathbb{R}^N)} + \|\phi\|_{L^1(\mathbb{R}^N)} - \int_0^t \Big[\big((\alpha-\delta-\xi)\big\|u(s)\big\|^2_{L^2(\mathbb{R}^N)} + \big\|\nabla u(s)\big\|^2_{L^2(\mathbb{R}^N)}\big) \\ &\quad\quad - \left(\frac{1}{4\xi}\|C_\delta\|^2_{L^2(\mathbb{R}^N)} + \|\psi\|_{L^1(\mathbb{R}^N)}\right)\Big]\,ds, \end{aligned} \tag{61}$$

which asymptotic estimate of the $H$-solutions will be concluded in a form of the lemma below.

**Lemma 5.** *For any $M > \frac{1}{4\xi}\|C_\delta\|^2_{L^2(\mathbb{R}^N)} + \|\psi\|_{L^1(\mathbb{R}^N)}$ and any bounded (satisfying $E(u_0)<\infty$) set $B$, there exists a time $T = T_{B,M}$ such that for any solution with initial data in $B$, there is a $t_0 \in [0,T]$ (depending on the initial data) such that*

$$\big\|\nabla u(t_0)\big\|^2_{L^2(\mathbb{R}^N)} + (\alpha-\delta-\xi)\big\|u(t_0)\big\|^2_{L^2(\mathbb{R}^N)} \leqslant M. \tag{62}$$

Define next the set

$$B_0 = \bigcup_{t\geqslant 0} S(t)B_1, \tag{63}$$

where

$$B_1 = \big\{u_0 \in H^1\big(\mathbb{R}^N\big)\colon\ \|\nabla u_0\|^2_{L^2(\mathbb{R}^N)} + (\alpha-\delta-\xi)\|u_0\|^2_{L^2(\mathbb{R}^N)} \leqslant M,\ \big\|(-\Delta)^{-\frac{1}{2}}u_0\big\|_{L^2(\mathbb{R}^N)} < +\infty\big\}. \tag{64}$$

We have

**Lemma 6.** *The set $B_0$ is bounded in $H^1(\mathbb{R}^N)$ and can be seen as a bounded absorbing set in $H^1(\mathbb{R}^N)$ for the semigroup of the $H$-solutions.*

**Proof.** The following properties of the sets $B_0, B_1$ can be easily seen:

- From (30) we know that $B_0$ is bounded in $H^1(\mathbb{R}^N)$.
- Obviously, $B_0$ is positively invariant, i.e., $S(t)B_0 \subset B_0$ for any $t \geqslant 0$.
- $B_1 \subset B_0$.
- From (62) we know that: for any bounded (satisfying $E(u_0) < \infty$) set $B$, there is a $T_B < \infty$ such that for any $u_0 \in B$, there exists (at least) some $t_0 \in [0, T]$ such that

$$S(t_0)u_0 \in B_1,$$

therefore

$$S(t_0)u_0 \in B_0.$$

Consequently, by the positive invariance of $B_0$

$$S(T_B)u_0 = S(T_B - t_0)S(t_0)u_0 \in B_0,$$

which implies that

$$S(T_B)B \subset B_0,$$

in particular,

$$S(t)B \subset B_0 \quad \text{for all } t \geqslant T_B.$$

Therefore, $B_0$ is a positively invariant bounded absorbing set in $H^1(\mathbb{R}^N)$. $\square$

Combining Lemma 6 with (16), we have immediately

**Theorem 1.** *Under the assumptions of Lemma* 5, *for the bounded absorbing set $B_0$ given in* (63), *there are positive constants $M$ and $T$* (*which depend on the $H^1$-bound of $B_0$*) *such that*

$$\left\| S(t)B_0 \right\|_{H^2(\mathbb{R}^N)} \leqslant M < \infty \quad \textit{for all } t \geqslant T.$$

We will formulate next a remark discussing a further regularity of the $H$-solutions for $t > 0$.

**Remark 6.** It was observed in Section 2.3 that under our Assumption II the (global) $H^1(\mathbb{R}^N)$-solutions to (2) enter $H^3(\mathbb{R}^N)$ for $t > 0$. This property gives also additional regularity of the $H$-solutions. For $t > 0$ the potential $\omega = \Delta u + f(x,u) - \nu u_t$ will belong to $H^1(\mathbb{R}^N)$. Next, due to the equality (40), $(1-\nu)(-\Delta)^{-1}u_t = \Delta u + f(x,u) - \nu u_t$, valid for the $H$-solutions, also $(-\Delta)^{-1}u_t \in H^1(\mathbb{R}^N)$ for $t > 0$.

A more complete discussion of the asymptotic behavior of the $H$-solutions will be given in the forthcoming publication. Now we recall the known in the literature properties of an operator $(-\Delta)^{-1}$.

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## Appendix A. The operator $(-\Delta)^{-1}$ in $\mathbb{R}^N$

As mentioned in the Introduction the operator $(-\Delta)^{-1}$ is often unbounded (of course, depending on the spaces it acts on). We describe here better its properties. First, using the Fourier transform, an explicit form of that operator can be found (see [7, p. 5]). Denoting $G_0(x) = c_N|x|^{-(N-2)}$, we have namely:

$$(-\Delta)^{-1}\phi = G_0 * \phi, \tag{A.1}$$

when $N > 2$. The properties of the operator (A.1) will be then studied based on the following ingenious Hardy–Littlewood–Sobolev inequality that can be found in [14, Chapter 4.5].

Denote $k_a(y) = |y|^{-\frac{N}{a}}$, where $y \in \mathbb{R}^N$, $a \in \mathbb{R}$. It is easy to see that $\int k(y)^r\,dy$ is divergent at infinity when $\frac{r}{a} \leqslant 1$, also it is divergent at 0 when $\frac{r}{a} \geqslant 1$. However, the following result is valid, just as this function will belong to $L^a(\mathbb{R}^N)$:

**Proposition 1.** *Let* $1 < a < \infty$, $1 < p < q < \infty$ *and*

$$\frac{1}{p} + \frac{1}{a} = 1 + \frac{1}{q}.$$

*Then*

$$\|k_a * \phi\|_{L^q(\mathbb{R}^N)} \leqslant C_{p,a}\|\phi\|_{L^p(\mathbb{R}^N)}, \tag{A.2}$$

*where* $\phi \in C^0(\mathbb{R}^N)$ *and* $k_a$ *is the kernel introduced above.*

Note, that in the case of the kernel $G_0$ introduced in (A.1), the corresponding number $a$ equals $\frac{N}{N-2}$ $(N > 2)$. A direct application of the proposition above gives us an estimate; since the relation between the constants reads $\frac{1}{p} = \frac{2}{N} + \frac{1}{q}$ in that case, for $q = \frac{2N}{N-2}$ we have that

$$\left\|(-\Delta)^{-1}\phi\right\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} = \|G_0 * \phi\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \leqslant c_N\|\phi\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}, \tag{A.3}$$

which shows that the $(-\Delta)^{-1}$ is a bounded operator from $L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ into $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ (and so, in particular, from $L^{\frac{6}{5}}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$).

Since the following condition

$$-\Delta u = 0, \quad u \in H^1(\mathbb{R}^N) \quad \Rightarrow \quad \int_{\mathbb{R}^N} |\nabla u|^2\,dx = 0, \quad u \in H^1(\mathbb{R}^N) \quad \Rightarrow \quad u = const = 0 \quad \text{a.e. in } \mathbb{R}^N,$$

then $(-\Delta): H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ is invertible. But the inverse $(-\Delta)^{-1}: H^{-1}(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is unbounded. Indeed, if it is bounded, then

$$\exists_{C_1>0}\ \forall_{\phi\in H^{-1}(\mathbb{R}^N)} \quad \left\|(-\Delta)^{-1}\phi\right\|_{H^1(\mathbb{R}^N)} \leqslant C_1\|\phi\|_{H^{-1}(\mathbb{R}^N)},$$

which can be extended to

$$C\left\|(-\Delta)^{-1}\phi\right\|_{L^2(\mathbb{R}^N)} \leqslant \left\|(-\Delta)^{-1}\phi\right\|_{H^1(\mathbb{R}^N)} \leqslant C_1\|\phi\|_{H^{-1}(\mathbb{R}^N)} \leqslant CC_1\|\phi\|_{L^2(\mathbb{R}^N)},$$

so that $(-\Delta)^{-1}$ has to be bounded in $L^2(\mathbb{R}^N)$, which is not the case.

Quoting [29, p. 86], we also have a slightly more general than (A.2) form of the Sobolev type inequality for Riesz potentials (covering our case when $\alpha = 2$). For $0 < \alpha < N$ denote by $I_\alpha$ the convolution:

$$I_\alpha * f(x) = c_N \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}\,dy. \tag{A.4}$$

The following estimate is valid:

**Proposition 2.** *Let $\alpha > 0$, $1 < p < \infty$, and $\alpha p < N$. Then there exists a constant $C = C(N, p)$ such that*

$$\|I_\alpha * f\|_{L^{p^*}(\mathbb{R}^N)} \leqslant C\|f\|_{L^p(\mathbb{R}^N)}, \quad p^* = \frac{Np}{N-\alpha p}, \tag{A.5}$$

*whenever $f \in L^p(\mathbb{R}^N)$.*

The Riesz potentials enjoy the following *additivity property* (see [19, Corollary 4.4]).

**Corollary 1.** *If* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ *and* $\operatorname{Re}(\alpha+\beta) < \frac{N}{2p}$, *then*

$$R_\alpha R_\beta \phi = R_{\alpha+\beta}\phi \tag{A.6}$$

*for all $\phi \in L^p(\mathbb{R}^N)$. Here $R_\alpha \phi = (-\Delta)^{-\alpha}\phi$ for all $\phi \in L^p(\mathbb{R}^N)$, $1 \leqslant p < \frac{N}{2\operatorname{Re}\alpha}$.*

The property (A.6), for elements of $\mathcal{S}$, was first reported in [25, Chapter V]:

$$R_\alpha R_\beta \phi = R_{\alpha+\beta}\phi, \quad \phi \in \mathcal{S},\ \alpha > 0,\ \beta > 0,\ \alpha+\beta < \frac{N}{2}.$$

The following estimates hold, which together with the denseness of $C_0^\infty(\mathbb{R}^N)$ in $W^{k,p}(\mathbb{R}^N)$ allows us to study the Lyapunov function (34) (even for the $H^1(\mathbb{R}^N)$-solutions).

**Lemma 7.** *The following property holds, for arbitrary function $\phi \in \mathcal{T}$,*

$$\int_{\mathbb{R}^N} (-\Delta)^{-1}\phi\phi\,dx \geqslant 0. \tag{A.7}$$

**Proof.** To justify the condition above we will use the results of [19]. For $0 < \alpha < \frac{N}{2}$ $(N \geqslant 3)$, and for $f$ locally integrable on $\mathbb{R}^N$, denote

$$R_\alpha f(x) = \frac{\Gamma(N/2-\alpha)}{\pi^{N/2}2^{2\alpha}\Gamma(\alpha)} \int_{\mathbb{R}^N} |x-y|^{2\alpha-N} f(y)\,dy = I_{2\alpha} * f(x)$$

the *Riesz potential of order* $\alpha$. Let $\mathcal{T}$ be the space of functions $\phi : \mathbb{R}^N \to \mathbb{C}$, of class $C^\infty$, such that any partial derivative of $\phi$ belongs to $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, endowed with the topology defined by the seminorms:

$$|\phi|_m = \max\bigl\{\bigl\|D^\beta\phi\bigr\|_{L^1(\mathbb{R}^N)}, \bigl\|D^\beta\phi\bigr\|_{L^\infty(\mathbb{R}^N)};\ |\beta| \leqslant m\bigr\}, \quad \phi \in \mathcal{T},\ m \in \mathbb{N}.$$

Evidently $\mathcal{S} \subset \mathcal{T} \subset L^p(\mathbb{R}^N)$ for arbitrary $1 \leqslant p \leqslant \infty$ (see [19] for more properties of $\mathcal{T}$).

If $\phi \in \mathcal{T} \subset L^{\frac{2N}{N+2}}(\mathbb{R}^N)$, then by (A.5) $R_{\frac{1}{2}}\phi \in L^2(\mathbb{R}^N)$. Using the additivity property of Riesz potentials, we have: for real $\phi \in \mathcal{T}$,

$$\int_{\mathbb{R}^N} (-\Delta)^{-1}\phi\phi\,dx = \int_{\mathbb{R}^N} R_1\phi\phi\,dx = \int_{\mathbb{R}^N} R_{\frac{1}{2}}R_{\frac{1}{2}}\phi\phi\,dx = \int_{\mathbb{R}^N} |R_{\frac{1}{2}}\phi|^2\,dx \geqslant 0, \tag{A.8}$$

the last equality following from the Tonelli–Hobson theorem, and we obtain the claim. □

**Remark 7.** Considerations similar to (A.8) in a more standard dense subset $\mathcal{S}$ of $L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ will be shown based on the properties of the Riesz potentials described in [25, Chapter V].

## References


[1] N.D. Alikakos, P. Bates, X. Chen, Convergence of the Cahn–Hilliard equation to the Hele-Shaw model, Arch. Ration. Mech. Anal. 128 (1994) 165–205.
[2] A. Bonfoh, Finite-dimensional attractor for the viscous Cahn–Hilliard equation in an unbounded domain, Quart. Appl. Math. 64 (2006) 93–104.
[3] L.A. Caffarelli, N.E. Muler, An $L^\infty$ bound for solutions of the Cahn–Hilliard equation, Arch. Ration. Mech. Anal. 133 (1995) 129–144.
[4] A.N. Carvalho, T. Dlotko, Dynamics of the viscous Cahn–Hilliard equation, J. Math. Anal. Appl. 344 (2008) 703–725.
[5] J.W. Cholewa, T. Dlotko, Global Attractors in Abstract Parabolic Problems, Cambridge University Press, Cambridge, 2000.
[6] T. Dlotko, C. Sun, Dynamics of the modified viscous Cahn–Hilliard equation in $\mathbb{R}^N$, Topol. Methods Nonlinear Anal. 35 (2010) 277–294.
[7] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge University Press, Cambridge, 1990.
[8] M. Efendiev, A. Miranville, S.V. Zelik, Exponential attractors for a singularly perturbed Cahn–Hilliard system, Math. Nachr. 272 (2004) 11–31.
[9] C.M. Elliott, A.M. Stuart, Viscous Cahn–Hilliard equation II. Analysis, J. Differential Equations 128 (1996) 387–414.
[10] P.C. Fife, Pattern formation in gradient systems, in: Handbook of Dynamical Systems, vol. 2. Applications, Elsevier, Amsterdam, 2002, pp. 677–722.
[11] H. Gajewski, K. Gröger, K. Zacharias, Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akademie-Verlag, Berlin, 1974.
[12] S. Gatti, M. Grasselli, A. Miranville, V. Pata, Hyperbolic relaxation of the viscous Cahn–Hilliard equation in 3-D, Math. Models Methods Appl. Sci. 15 (2005) 165–198.
[13] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin, 1981.
[14] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer-Verlag, Berlin, 1983.
[15] M.B. Kania, Global attractor for the perturbed viscous Cahn–Hilliard equation, Colloq. Math. 109 (2007) 217–229.
[16] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1980.
[17] H. Komatsu, Fractional powers of operators, Pacific J. Math. 19 (1966) 285–346.
[18] S. Liu, F. Wang, H. Zhao, Global existence and asymptotics of solutions of the Cahn–Hilliard equation, J. Differential Equations 238 (2007) 426–469.
[19] C. Martínez, M. Sanz, F. Periago, Distributional fractional powers of the Laplacean. Riesz potentials, Studia Math. 135 (1999) 253–271.
[20] A. Miranville, Some generalization of the Cahn–Hilliard equation, Asymptot. Anal. 22 (2000) 235–259.
[21] A. Novick-Cohen, On the viscous Cahn–Hilliard equation, in: J.M. Ball (Ed.), Material Instabilities in Continuum Mechanics and Related Mathematical Problems, Clarendon Press, Oxford, 1988, pp. 329–342.
[22] M. Polat, A.O. Çelabi, N. Çelabi, Global attractors for the 3D viscous Cahn–Hilliard equations in an unbounded domain, Appl. Anal. 88 (2009) 1157–1171.
[23] G. Raugel, Global attractors in partial differential equations, in: B. Fiedler (Ed.), Handbook of Dynamical Systems, vol. 2, Elsevier, Amsterdam, 2002, pp. 885–982.
[24] A. Rodriguez-Bernal, A. Vidal-López, Semistable extremal ground states for nonlinear evolution equations in unbounded domains, J. Math. Anal. Appl. 338 (2008) 675–694.
[25] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1970.
[26] Ph. Souplet, Decay of heat semigroups in $L^\infty$ and applications to nonlinear parabolic problems in unbounded domains, J. Funct. Anal. 173 (2000) 343–360.
[27] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1997.
[28] S. Zheng, A. Milani, Global attractors for singular perturbations of the Cahn–Hilliard equations, J. Differential Equations 209 (2005) 101–139.
[29] W.P. Ziemer, Weakly Differentiable Functions, Sobolev Spaces and Functions of Bounded Variation, Springer-Verlag, New York, 1989.